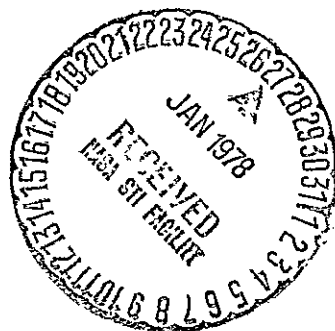


*CR 151602*

(NASA-CR-151602)    AN ANALYTICAL STATE TRANSITION MATRIX FOR ORBITS PERTURBED BY AN OBLATE SPHEROID (Analytical and Computational Mathematics, Inc.)    72 p HC A04/MF A01	N78-15137  Unclas 57825
---	----------------------------------

AN ANALYTICAL STATE  
TRANSITION MATRIX FOR ORBITS  
PERTURBED BY AN OBLATE SPHEROID

**A**NALYTICAL    AND  
**C**OMPUTATIONAL  
**M**ATHEMATICS, INC.



AN ANALYTICAL STATE TRANSITION MATRIX FOR ORBITS  
PERTURBED BY AN OBLATE SPHEROID

ALAN MUELLER

ANALYTICAL AND COMPUTATIONAL  
MATHEMATICS, INC.

1275 SPACE PARK DRIVE  
SUITE 114  
HOUSTON, TEXAS 77058  
713-333-3660

MAY 1977

This report was prepared for the NASA/Johnson Space Center  
under Contract NAS9-15171.

## CONTENTS

Section		Page
1.0	INTRODUCTION	9.
2.0	THE PS ELEMENT FORMULATION AND SATELLITE THEORY	12
	2.1 The $J_2$ Satellite Theory	13
3.0	THE STATE TRANSITION MATRIX	16
	3.1 The PS State Transition Matrix	17
	3.2 The PS State Transition Matrix in Time	20
4.0	CARTESIAN TRANSITION MATRIX	22
5.0	NUMERICAL EXPERIMENTS	25
6.0	CONCLUSIONS	30
	REFERENCES	31
	APPENDIX A	35
	APPENDIX B	43
	APPENDIX C	53
	APPENDIX D	67

## TABLES

Table		Page
I	ORBIT TYPES	26

## FIGURES

Figure			Page
1	Matrix Accuracy	Case 1	27
2	Matrix Accuracy	Case 2	28
3	Matrix Accuracy	Case 3	29

AN ANALYTICAL STATE TRANSITION MATRIX FOR ORBITS  
PERTURBED BY AN OBLATE SPHEROID

by  
Alan Mueller

1.0 Introduction

Often in orbit determination or navigation algorithms one must predict how small deviations from some nominal orbit will in time cause the actual orbital path to veer from the nominal path. If the initial deviation is small, then a linear approximation may be used to determine the deviation at any given time. The linear approximation, a truncated Taylor series expansion, requires the first partial derivatives of the coordinates defining the position and velocity at a given time, with respect to the coordinates defining the position and velocity at the initial time. This matrix of partial derivatives is commonly called the state transition matrix -  $\Phi$

$\Phi$  is governed by the matrix differential equation given by Battin (Reference 1).

$$\dot{\Phi} = G \cdot \Phi \quad (1.1)$$

where  $G$  is the matrix of partial derivatives of the rate of change of the position and velocity. Initially,  $\Phi$  is equal to the identity matrix.  $\Phi$  may be determined by numerically integrating the above differential equation. But when one considers the fact that the transition matrix is usually used in some iterative manner, the computational costs required by the numerical integration of the matrix differential equation becomes prohibitive.

The alternative, of course, is to determine the transition matrix by some analytical technique. If the final state (position and velocity)  $x_f$  is given as a function of the initial state  $x_o$  and the final time  $t_f$

$$x_f = x_f(x_o(t_o), t_f) \quad (1.2)$$

then  $\Phi$  may be determined by taking partial derivatives of this functional relation

$$\Phi(t_f, t_o) = \frac{\partial x_f}{\partial x_o}(x_o(t_o), t_f) \quad (1.3)$$

Several analytical techniques (References 2, 3 and 4) have determined expressions for the transition matrix under the assumption that the satellite moves along a two-body orbit, all perturbations being neglected. But as Rice, (Reference 5) has pointed out, this assumption may result in non-negligible errors if the perturbations are large.

For artificial satellites orbiting near the earth, the  $J_2$  oblateness potential contributes a strong perturbation which may not be neglected for accurate satellite orbit prediction. A first order, canonical analytical theory by Brouwer (Reference 6) and rewritten in non-singular elements by Lyddane (Reference 7) does account for the oblateness perturbation. However, the complexity of the theory makes it somewhat cumbersome to use as a basis for developing an analytical  $J_2$  transition matrix. However, significant advantages are offered by a  $J_2$  satellite theory developed from a new set of canonical elements proposed by Scheifele (Reference 8). The elements are in an extended phase space, in that eight (instead of six) variables describe the state. They also have, as their independent variable, a quantity related to the true anomaly instead of time. The set is similar to

the Poincare elements and are therefore named the Poincare-Similar (PS) elements. The PS elements are non-singular for bounded orbits.

Recently, the elements have been applied to both numerical and analytical orbit prediction. The expressions for converting to and from the PS and Cartesian state and the expressions for the equations of motion are given by Mueller (Reference 9). The PS elements have proven to be a very accurate and stable set for numerical integration (Reference 10). And as was said, the PS elements have been applied very successfully to the analytical prediction of orbits perturbed by oblateness in a theory by Bond and Scheifele (Reference 11).

Two aspects of the PS elements allow for a very simple but accurate satellite theory. Because one of the PS variables is associated with the true anomaly, the Hamiltonian of the zonal oblateness problem becomes a finite expression in the elements. The result is a more concise satellite theory. The other aspect is the appearance of the total energy (instead of the osculating two-body energy) as a canonical variable which describes the mean motion. Since a "second integration" or a different kind of higher order precision is unnecessary, the initialization procedure to find the "mean elements" is straight forward and no iterative procedure is required. The result is a considerable improvement in the accuracy.

Because of the simplicity and accuracy of the canonical analytical solution based on PS elements, it becomes a logical choice on which to base a new analytical state transition matrix. The intention of this paper therefore, is to derive a completely analytical singularity free form of the state transition matrix for orbits perturbed by an oblate spheroid.



## 2.0 The PS Element Formulation and Satellite Theory

The derivation of the PS elements and their corresponding  $J_2$  satellite theory have been described in detail in (References 8, 9, and 11). A short description of the elements and the satellite theory will be given here with all necessary equations listed in Appendices A and B.

Expressions in Appendix A describe the relations between the Cartesian state vector and time

$$\mathbf{x} = (x_1, x_2, x_3, x_4 = \dot{x}_1, x_5 = \dot{x}_2, x_6 = \dot{x}_3)$$

$$t = \text{time}$$

and the PS elements

$$\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5 = \rho_1, \sigma_6 = \rho_1, \sigma_7 = \rho_3, \sigma_8 = \rho_4)$$

The relation between the time  $t$  and the new independent variable  $\tau$  is given by

$$\frac{dt}{d\tau} = \frac{r^2}{q} \quad (2.1)$$

where  $r$  is the position magnitude and  $q$  is given by

$$q = -\frac{1}{2} (\sigma_2^2 + \rho_2^2 - \rho_1 - \frac{\mu}{\sqrt{2\rho_4}})$$

The hamiltonian describing the motion of a perturbed satellite is

$$F = \rho_1 - \frac{\mu}{\sqrt{2\rho_4}} + \epsilon F_1 \quad (2.2)$$

where

$$\epsilon F_1 = \frac{r^2}{q} V \quad (V = \text{perturbing potential})$$

The canonical differential equations are expressed as

$$\frac{d\sigma_j}{d\tau} = \frac{\partial F}{\partial \rho_j}, \quad \frac{d\rho_j}{d\tau} = - \frac{\partial F}{\partial \sigma_j}, \quad (j=1,2,3,4) \quad (2.3)$$

and thus for unperturbed motion ( $V=0$ ) the solution of these equations is simply

$$\begin{aligned} \sigma_1 &= \tau + \sigma_{10} \\ \sigma_4 &= \frac{\mu}{(2\rho_{40})^{3/2}} \tau + \sigma_{40} \end{aligned} \quad (2.4)$$

All other elements are constants in the unperturbed case.

## 2.1 The $J_2$ Satellite Theory

A complete first order solution of the motion of a satellite perturbed by oblateness has been developed by Bond and Scheifele (Reference 11). A brief outline of the solution will be given here.

The hamiltonian for the  $J_2$  perturbed case now reads

$$F = \rho_1 - \frac{\mu}{\sqrt{2\rho_4}} + \epsilon F_1 \quad (2.5)$$

where

$$F_1 = \frac{1}{r} \left[ \left( \frac{x_3}{r} \right)^2 - \frac{1}{3} \right] \quad (2.6)$$

and 
$$\varepsilon = \frac{3}{2} J_2 \mu R^2$$

$R$  = mean equatorial radius

$\mu$  = Gravitational constant

$J_2$  = oblateness coefficient.

The differential equations are solved by a method of Von-Zeipel. The elements undergo a canonical transformation thru a determining function  $S_1$  so that the short periodic terms are eliminated from the hamiltonian. The equations of motion in the transformed system  $\sigma'$  may then be solved with an accuracy of order  $\varepsilon$ .

The solution algorithm can be divided into three steps:

1. Canonical transformation to eliminate short periodic terms:

$$\sigma'_{ko} = \sigma_{ko} + \varepsilon \frac{\partial S_1}{\partial \rho_{ko}} (\sigma_o, \rho_o) \quad k=1,2,3,4 \quad (2.7)$$

$$\rho'_{ko} = \rho_{ko} - \varepsilon \frac{\partial S_1}{\partial \sigma_{ko}} (\sigma_o, \rho_o)$$

2. The analytical integration of the transformed equations of motion:

$$\sigma'_1 = \sigma'_{10} + A_1 \tau$$

$$\sigma'_2 = \sigma'_{20} \cos(A_2 \tau) - \rho'_{20} \sin(A_2 \tau)$$

$$\sigma'_3 = \sigma'_{30} \cos(A_3 \tau) - \rho'_{30} \sin(A_3 \tau)$$

$$\sigma'_4 = \sigma'_{40} + A_4 \tau \quad (2.8)$$

$$\rho'_1 = \rho'_{10}$$

$$\rho'_2 = \rho'_{20} \cos (A_2 \tau) + \sigma'_{20} \sin (A_2 \tau)$$

$$\rho'_3 = \rho'_{30} \cos (A_3 \tau) + \sigma'_{30} \sin (A_3 \tau)$$

$$\rho'_4 = \rho'_{40}$$

3. The back transformation:

$$\begin{aligned} \sigma_k &= \sigma'_k - \varepsilon \frac{\partial S_1}{\partial \rho'_k} (\sigma', \rho') \\ \rho_k &= \rho'_k + \varepsilon \frac{\partial S_1}{\partial \sigma'_k} (\sigma', \rho') \end{aligned} \quad (2.9)$$

The partial derivatives of the determining function  $S_1$  and the expressions for  $A_1$ ,  $A_2$ ,  $A_3$  and  $A_4$  may be found in Appendix B.

The relation which gives the back transformation in Step 3 is simply the inverse of the relation in Step 1, and has an accuracy of order  $\varepsilon$

If one wishes to determine the state  $\sigma$  as a function of a specific time, then an iteration procedure is required which is similar to the iteration to solve Keplers equation in classical theory. A regula-falsi method or a tangent method using equation (2.1) may be used to solve the non-linear equation. The algorithm for the entire procedure is then

1. given  $\sigma (\tau=0) \xrightarrow{(2.7)} \sigma' (\tau=0)$
2. given  $\sigma' (\tau=0)$  and  $\tau_i \xrightarrow{(2.8)} \sigma' (\tau_i)$
3. given  $\sigma' (\tau_i) \xrightarrow{(2.9)} \sigma (\tau_i)$
4. given  $\sigma (\tau_i) \xrightarrow{(A.12)} t_i$
5. if  $|t_i - t| < \text{tol} \rightarrow \text{stop}$
6. given  $t_i \xrightarrow{(2.1)} \tau_{i+1}$
7. go to 2

The number above the arrow indicates the equation by which the calculation is made. Tol is the maximum error tolerable in the iteration scheme.

### 3.0 The State Transition Matrix

Suppose that the final and initial states are defined in the PS space as  $\sigma_f$  and  $\sigma_o$ . Then the PS transition matrix  $T$  is defined as

$$T (t_f, t_o) = \frac{\partial \sigma_f}{\partial \sigma_o} (t_f, t_o) \quad (3.1)$$

The Cartesian transition matrix  $\Phi$  defined in equation (1.3) may be determined from  $T$  by the chain rule

$$\Phi (t_f, t_o) = \frac{\partial x_f}{\partial \sigma_f} T \frac{\partial \sigma_o}{\partial x_o} \quad (3.2)$$

The development of  $T$  is described fully in Section 3.1 while the partial derivatives of the Cartesian and PS state are derived in Section 3.2.

### 3.1 The PS State Transition Matrix

Since the PS variables have as their independent variable a quantity related to the true anomaly  $\tau$ , it is convenient to first develop a transition matrix which is a function of the initial and final values of the independent variable.

$$T^* (\tau_f, \tau_o) = \frac{\partial \sigma_f}{\partial \sigma_o} (\tau_f, \tau_o) \quad (3.3)$$

However the matrix  $T^*$  in no way equals the matrix  $T$  defined by equation (3.1) for reasons to be made clear in Section 3.2.

Like in the analytical solution which is obtained in three steps, the matrix  $T^*$  can be expressed as a product of three matrices

$$T^* = T_1 \cdot T_2^* \cdot T_3 \quad (3.4)$$

where

$$\left. \begin{aligned} T_1 (\sigma'(\tau_f)) &= \frac{\partial \sigma(\tau_f)}{\partial \sigma'(\tau_f)} \\ T_2^* (\sigma'(\tau_o), \tau_f) &= \frac{\partial \sigma'(\tau_f)^*}{\partial \sigma'(\tau_o)} \\ T_3 (\sigma(\tau_o)) &= \frac{\partial \sigma'(\tau_o)}{\partial \sigma(\tau_o)} \end{aligned} \right\} \quad (3.5)$$

The values of  $\sigma'$  which are used in the corresponding matrices may be obtained from the analytical solution. Since the partial derivatives for the matrix elements of  $T_1$ ,  $T_2^*$  and  $T_3$  are to be derived from the differentiation of the analytical solution, it will be accurate up to terms of order  $O(\epsilon^2)$  only. Therefore, it is adequate to evaluate

the matrices with values obtained from the analytical solution and not a numerically computed nominal solution. However, it is possible to insert either an analytically or numerically integrated solution which includes other perturbations, such as drag or higher harmonics. This may result in a more accurate model and is a possible area for research.

By denoting an element of a matrix as

$$\left[ T^* \right]_{kj} = \frac{\partial \sigma_k(\tau_f)}{\partial \sigma_j(\tau_o)}$$

then each element of the three may be deduced from equations (2.7), (2.8) and (2.9), and expressed as

for  $n = 1, 2, \dots, 8$

$$\left. \begin{aligned} \left[ T_1 \right]_{kn} &= \psi_{kn} + \epsilon \frac{\partial^2 S_1}{\partial \sigma'_{k+4} \partial \sigma'_n} \\ \left[ T_1 \right]_{k+4,n} &= \psi_{k+4,n} - \epsilon \frac{\partial^2 S_1}{\partial \sigma'_k \partial \sigma'_n} \end{aligned} \right\} \begin{array}{l} (k=1, 2, 3, 4) \\ (3.6) \end{array}$$

for  $j = 1, 2, \dots, 8$

$$\left. \begin{aligned} \left[ T_3 \right]_{mj} &= \psi_{mj} - \epsilon \frac{\partial^2 S_1}{\partial \sigma_{m+4} \partial \sigma_j} \\ \left[ T_3 \right]_{m+4,j} &= \psi_{m+4,j} + \epsilon \frac{\partial^2 S_1}{\partial \sigma_m \partial \sigma_j} \end{aligned} \right\} \begin{array}{l} (m=1, 2, 3, 4) \\ (3.7) \end{array}$$

for  $m = 1, 2, \dots, 8$

$$\begin{aligned}
 \left[ T_1^* \right]_{1m} &= \tau A_{1m} + \psi_{1m} \\
 \left[ T_2^* \right]_{2m} &= -\sigma_6 \tau A_{2m} + \psi_{2m} \cos(A_2 \tau) \\
 &\quad - \psi_{6m} \sin(A_2 \tau) \\
 \left[ T_2^* \right]_{3m} &= -\sigma_7 \tau A_{3m} + \psi_{3m} \cos(A_3 \tau) \\
 &\quad - \psi_{7m} \sin(A_3 \tau) \\
 \left[ T_2^* \right]_{4m} &= \tau A_{4m} + \psi_{4m} \\
 \left[ T_5^* \right]_{5m} &= \psi_{5m} \\
 \left[ T_2^* \right]_{6m} &= \sigma_2 \tau A_{2m} + \psi_{6m} \cos(A_2 \tau) \\
 &\quad + \psi_{2m} \sin(A_2 \tau) \\
 \left[ T_2^* \right]_{7m} &= \sigma_3 \tau A_{3m} + \psi_{7m} \cos(A_3 \tau) \\
 &\quad + \psi_{3m} \sin(A_3 \tau) \\
 \left[ T_2^* \right]_{8m} &= \psi_{8m}
 \end{aligned} \tag{3.8}$$

where  $\psi_{kj}$  is each element of the identity matrix. The partial derivatives of  $S_1$  and  $A_{nm}$  can be found in Appendix C.



### 3.2 The PS Transition Matrix in Time

In most applications of the transition matrix, it is preferable to deal with a matrix of the form

$$T(t_f, t_o) \quad (3.9)$$

instead of

$$T^*(\tau_f, \tau_o) \quad (3.10)$$

The subsequent considerations are to be done for any analytical transition matrix whether perturbed or unperturbed or whether time or some other variable is used as the independent variable. The reason is that no satellite theory can be expressed as a closed form function of time. In classical theory one must introduce the eccentric anomaly. Similarly, a new independent variable, the true anomaly, is introduced in the PS formulation.

From equations (2.7), (2.8) and (2.9) one may write the solution in functional relations as<sup>†</sup>

$$\left. \begin{aligned} \sigma'_o &= \sigma'_o(\sigma_o) \\ \sigma' &= \sigma'(\sigma'_o, \tau) \\ \sigma &= \sigma(\sigma') \end{aligned} \right\} \quad (3.11)$$

which are used to compute the PS transition matrix

$$T = \frac{\partial \sigma}{\partial \sigma_o} = \frac{\partial \sigma}{\partial \sigma'} \frac{\partial \sigma'}{\partial \sigma'_o} \frac{\partial \sigma'_o}{\partial \sigma_o}$$

or

$$T = T_1 \cdot T_2 \cdot T_3 \quad (3.12)$$

---

<sup>†</sup>  $\sigma$   $\sigma'$  are vectors

If  $T$  is to be of the form defined in (3.9) then  $T_2$  must be defined as follows

$$T_2 = \frac{\partial \sigma'}{\partial \sigma'_0} = \left( \frac{\partial \sigma'}{\partial \sigma'_0} \right)^* + \frac{\partial \sigma'}{\partial \tau} \frac{\partial \tau}{\partial \sigma'_0}$$

or

$$T_2 = T_2^* + \frac{\partial \sigma'}{\partial \tau} \frac{\partial \tau}{\partial \sigma'_0} \quad (3.13)$$

The additional term  $\frac{\partial \sigma'}{\partial \tau} \frac{\partial \tau}{\partial \sigma'_0}$  results from the property that the end value of the independent variable  $\tau$  is now determined from the given value of the final time and initial PS vector  $\sigma_0$ . However, there is no explicit function for  $\tau$  in terms of the variables  $\sigma'_0$ . Thus one must determine the values of the vector  $\frac{\partial \tau}{\partial \sigma'_0}$  in a manner which is very similar to the method in which the partial derivatives of the eccentric anomaly in classical theory are determined (Reference 2).

Since now the final time is to be a given value, it is not determined by the initial conditions  $\sigma_0$ . Therefore,

$$\frac{\partial t}{\partial \sigma_0} = 0 \quad \text{or} \quad \frac{\partial t}{\partial \sigma'_0} = 0 \quad (3.14)$$

By the chain rule

$$\frac{\partial t}{\partial \sigma'_0} = \frac{\partial t}{\partial \sigma} \frac{\partial \sigma}{\partial \sigma'} \left( \frac{\partial \sigma'}{\partial \sigma'_0} \right)^* + \frac{\partial t}{\partial \sigma} \frac{\partial \sigma}{\partial \sigma'} \frac{\partial \sigma'}{\partial \tau} \frac{\partial \tau}{\partial \sigma'_0} = 0$$

from which one may deduce

$$\frac{\partial \tau}{\partial \sigma'_0} = - \frac{\partial t}{\partial \sigma} \frac{\partial \sigma}{\partial \sigma'} \left( \frac{\partial \sigma'}{\partial \sigma'_0} \right)^* \left( \frac{\partial t}{\partial \sigma} \frac{\partial \sigma}{\partial \sigma'} \frac{\partial \sigma'}{\partial \tau} \right)^{-1} \quad (3.15)$$

According to the definition of the total derivative and since time is an implicit function of the independent variable  $\tau$ ,

$$\frac{dt}{d\tau} = \frac{\partial t}{\partial \sigma} \frac{\partial \sigma}{\partial \sigma'} \frac{\partial \sigma'}{\partial \tau} \quad (3.16)$$

Observing the definitions of  $T_1^*$  and  $T_2$  in equation (3.5) one obtains a concise expression for the desired vector

$$\frac{\partial \tau}{\partial \sigma'_0} = - \frac{\partial t}{\partial \sigma} T_1 T_2^* \left( \frac{dt}{d\tau} \right)^{-1} \quad (3.17)$$

Thus from equation (3.13) the expression for  $T_2$  is

$$T_2 = T_2^* - \frac{\partial \sigma'}{\partial \tau} \frac{\partial t}{\partial \sigma} T_1 T_2^* \left( \frac{dt}{d\tau} \right)^{-1} \quad (3.18)$$

The expressions for  $\frac{\partial \sigma'}{\partial \tau}$  may be found in Appendix C. While those for  $\frac{\partial t}{\partial \sigma}$  can be found in Appendix D.

To summarize, one can use expressions in equation (3.6), (3.7), (3.8), (3.12) and (3.18) to compute the  $J_2$  state transition matrix in the PS element system.

#### 4.0 Cartesian Transition Matrix

From equation (3.2) one knows that the Cartesian transition matrix  $\Phi$  may be determined from the PS matrix  $T$  by

$$\Phi = \frac{\partial x_f}{\partial \sigma_f} T \frac{\partial \sigma_o}{\partial x_o}$$

This leads to a delicate situation in which one must transform an 8x8 matrix to the 6x6 Cartesian matrix. To

ease the transformation, introduce an extended Cartesian space  $y$ . This state not only includes the six Cartesian position and velocity coordinates, but also the time and the total energy.

$$y = (y_1=x_1, \quad y_2=x_2, \quad y_3=x_3, \quad y_4= \text{time}, \\ y_5=x_4, \quad y_6=x_5, \quad y_7=x_6, \quad y_8= \text{energy})$$

An  $8 \times 8$  extended Cartesian matrix,  $U$ , can be determined by chain rule as

$$U = \frac{\partial y_f}{\partial \sigma_f}^T \frac{\partial \sigma_o}{\partial y_o} \quad (4.1)$$

As stated before, the Cartesian state and time are given as a function of the PS elements. In addition, the canonical transformation from  $y$  space to  $\sigma$  space requires that

$$\sigma_8 = y_8 \quad (4.2)$$

If one defines the matrix

$$W = \frac{\partial y}{\partial \sigma} \quad (4.3)$$

then  $W$  is a Jacobian of a canonical transformation and thus is a symplectic matrix obeying the following inverse relation

$$W^{-1} = \frac{\partial \sigma}{\partial y} = - J W^T J \quad (4.4)$$

where  $( )^T$  means transpose and  $J$  satisfies the relation

$$J^2 = - I$$

Thus by using canonical elements one gains an advantage in that only expressions for  $W$  need be determined. It's inverse may then be found by rearrangement of elements inside the matrix. The expressions for the matrix  $\frac{\partial y}{\partial \sigma}$  can be found in Appendix D.

Finally, if one observes the fact that the energy  $-y_8$  can be expressed as a function of the Cartesian state  $x$ .

$$y_8 = \frac{1}{2} (x_4^2 + x_5^2 + x_6^2) + \frac{\mu}{r} + V(x) \quad (4.5)$$

where  $V$  is the perturbing  $J_2$  potential, then  $\Phi$  may be expressed in terms of the extended matrix  $U$  by the relations

for  $n=1,2,3$   
 $m=1,2,3$

$$\begin{aligned} \begin{bmatrix} \Phi \end{bmatrix}_{n,m} &= \begin{bmatrix} U \end{bmatrix}_{n,m} + \begin{bmatrix} U \end{bmatrix}_{n,8} \frac{\partial y_8}{\partial x_m} \\ \begin{bmatrix} \Phi \end{bmatrix}_{n,m+3} &= \begin{bmatrix} U \end{bmatrix}_{n,m+4} + \begin{bmatrix} U \end{bmatrix}_{n,8} \frac{\partial y_8}{\partial x_{m+3}} \\ \begin{bmatrix} \Phi \end{bmatrix}_{n+3,m} &= \begin{bmatrix} U \end{bmatrix}_{n+4,m} + \begin{bmatrix} U \end{bmatrix}_{n+4,8} \frac{\partial y_8}{\partial x_m} \\ \begin{bmatrix} \Phi \end{bmatrix}_{n+3,m+3} &= \begin{bmatrix} U \end{bmatrix}_{n+4,m+4} + \begin{bmatrix} U \end{bmatrix}_{n+4,8} \frac{\partial y_8}{\partial x_{m+3}} \end{aligned} \quad (4.6)$$

where

$$\frac{\partial y_8}{\partial x_i} = -\frac{\mu}{r} x_i + \frac{\partial V}{\partial x_i} \quad i=1,2,3$$

$$\frac{\partial y_8}{\partial x_{i+3}} = x_{i+3}$$

The inverse of  $\Phi$  may be determined by observing that it too is a Jacobian of a canonical transformation. Therefore

$$\Phi^{-1} = - J \Phi^T J . \quad (4.7)$$

## 5.0 Numerical Experiments

Direct comparisons of the analytical transition matrix have been made to a transition matrix which is obtained through numerical integration of the matrix differential equations (1.1). These equations include the  $J_2$  perturbation. To demonstrate the accuracy advantages of the  $J_2$  transition matrix, the simple conic matrix<sup>†</sup> is also compared to the  $J_2$  numerically integrated matrix.

Accuracy of the analytical matrices is determined by  $\delta$  defined as

$$\delta = \frac{\sum_{K=1}^6 \sum_{L=1}^6 (A_{KL} - N_{KL})^2}{\sum_{K=1}^6 \sum_{L=1}^6 N_{KL}^2}$$

where

$A_{KL}$  is the K,L component of the analytical matrix

$N_{KL}$  is the K,L component of the numerical matrix

Both A and N matrices are in normalized units so that the summation makes sense.

Three orbits have been chosen for test cases. The initial conditions of each orbit are listed in Table I. Figures 1 through 3 give the time history of  $\delta$  over about 3 revolutions. On each figure appears two curves.

---

<sup>†</sup> Formulas for the conic transition matrix elements do not contain oblateness ( $J_2$ ) effects.

One curve is the time history of  $\delta$  using the  $J_2$  analytical matrix and the other is the time history of  $\delta$  using the simple conic matrix. Note that  $-\log\delta$  is plotted with time and is representative of the number of accurate decimal digits. Curves which appear to the top are therefore more accurate.

In all three cases the conic matrix degrades to less than two digits of accuracy within two revolutions whereas the  $J_2$  matrix maintains four to six digits of accuracy. By comparing cases 1 and 2, one finds no degradation of accuracy for circular and equatorial orbits.

TABLE I. (ORBIT TYPES)

	a(km)	e	I	$\Omega$	$\omega$	M
Case 1	6677.7	0.015	$30^\circ$	$0^\circ$	$0^\circ$	$20^\circ$
Case 2	6677.7	0.0	$0^\circ$	$0^\circ$	$0^\circ$	$20^\circ$
Case 3	13266.2	0.5	$0.0^\circ$	$0^\circ$	$0^\circ$	$20^\circ$

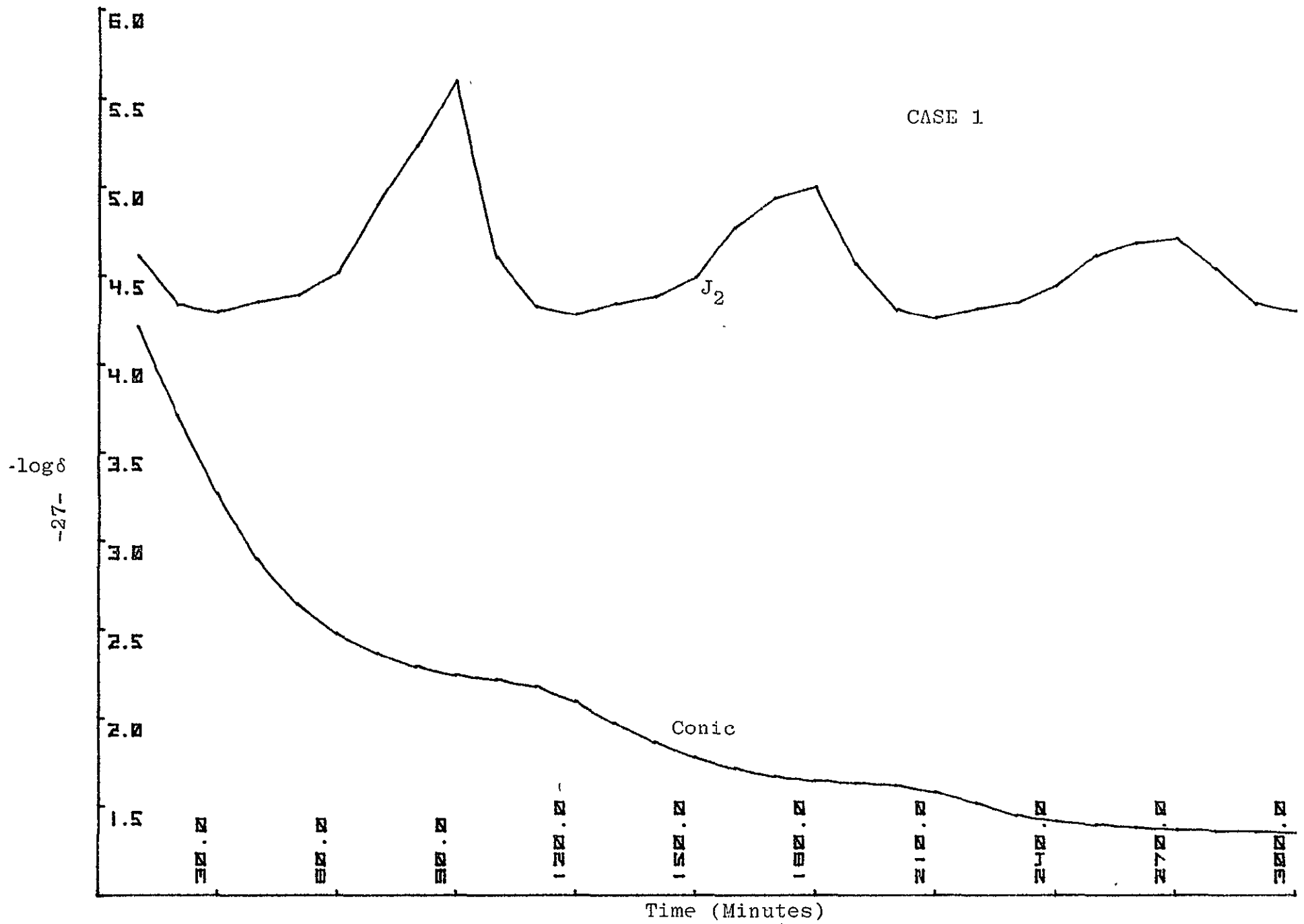
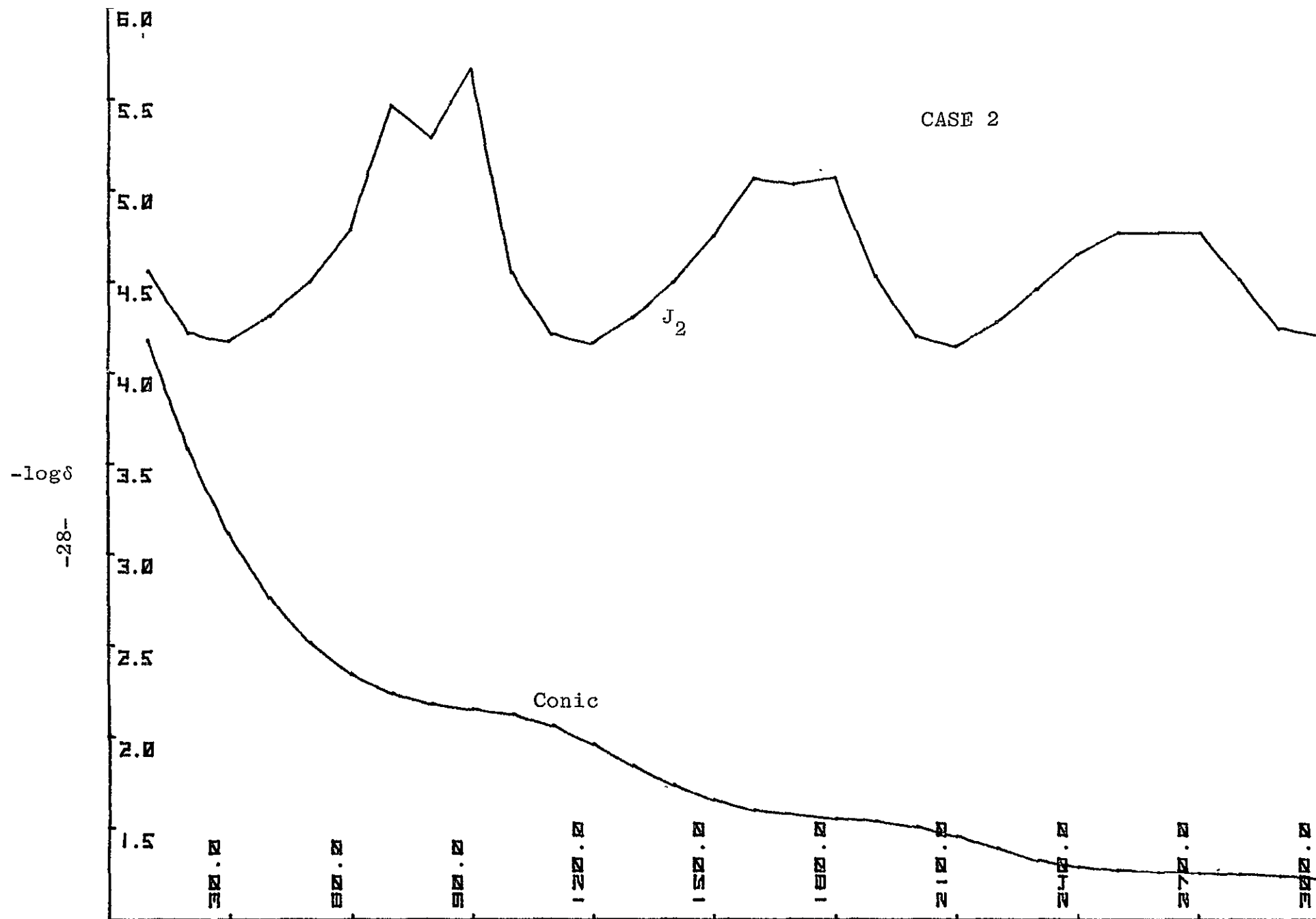


Figure 1  
MATRIX ACCURACY

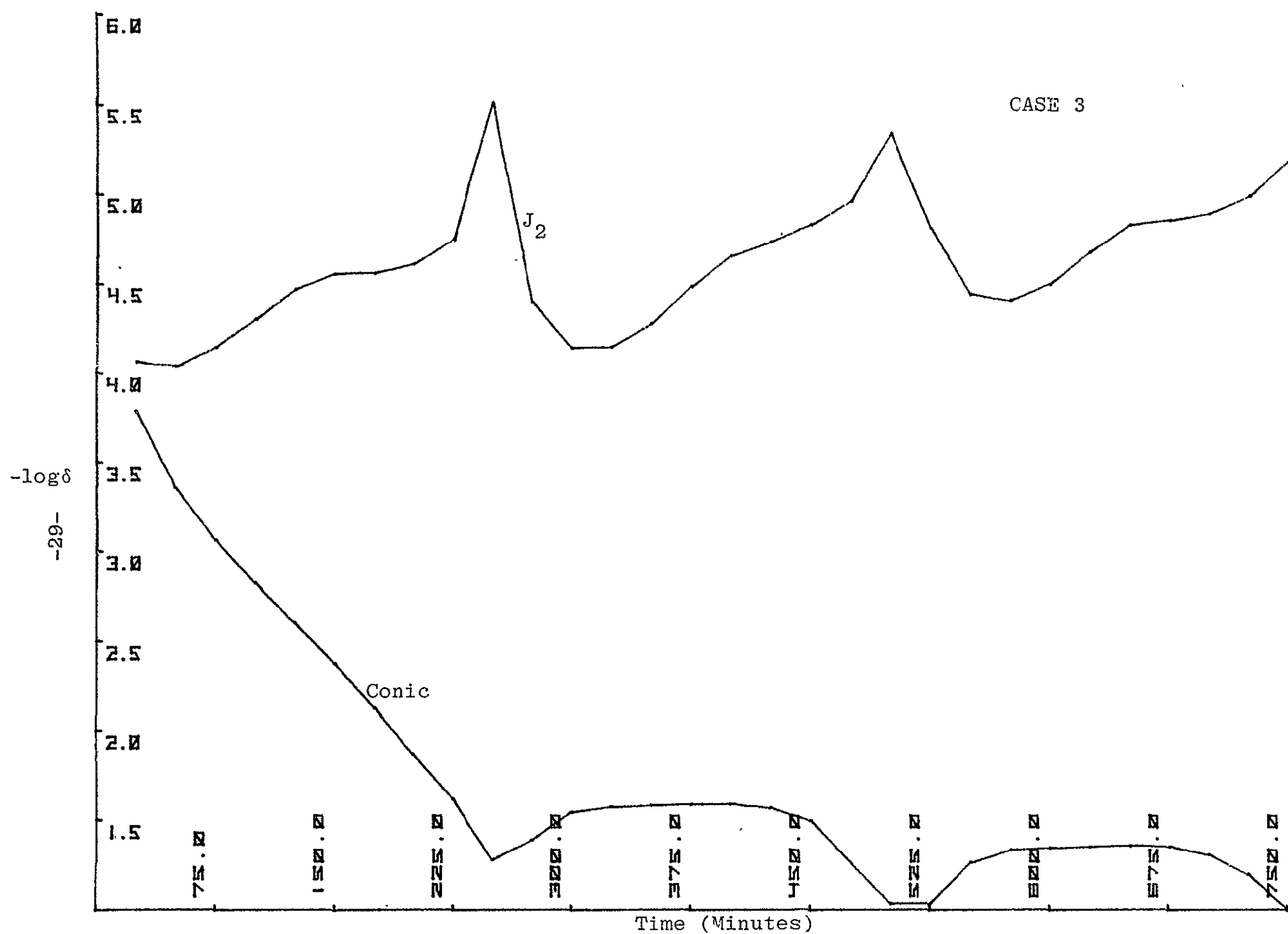




Time (Minutes)

Figure 2

MATRIX ACCURACY



Time (Minutes)

Figure 3

MATRIX ACCURACY

## 6.0 Conclusions

An analytical state transition matrix and its inverse, which include the short period and secular effects of the second zonal harmonic, has been developed from the non-singular PS satellite theory. The fact that the independent variable in the PS theory is not the time is in no respect disadvantageous, since any explicit analytical solution must be expressed in the true or eccentric anomaly. This is even the case for the simple conic matrix. The PS theory allows for a concise, accurate, and algorithmically simple state transition matrix. The improvement over the conic matrix range from 2 to 4 digits better accuracy..

REFERENCES

1. Battin, R.H.: Astronautical Guidance, McGraw-Hill, Inc. 1964.
2. Goodyear, W.H.: Completely General Closed-Form Solution for Coordinates and Partial Derivatives of the Two-Body Problem, *Astronomical Journal*, Volume 70, Number 3, April, 1965.
3. Danby, J.M.A.: The Matrizant of Keplerian Motion, *AIAA Journal*, Volume 2, No. 1, January, 1964.
4. Broucke, R.A.: On the Matrizant of the Two-Body Problem, *Astronautics and Astrophysics* 6, 1970.
5. Rice, D.R.: An Investigation into the Effects of Using Simplified Force in the Computation of State Transition Matrices, *AIAA Paper* 67-123, New York, 1967.
6. Brouwer, D.: Solution of the Problem of Artificial Satellite Theory Without Drag, *Astronomical Journal*, Volume 64, pp. 378-397.
7. Lyddane, H.: Small Eccentricities and Inclinations in the Brouwer Theory of the Satellite Problem, *Astronomical Journal*, Volume 68, pp. 555, 1963.
8. Scheifele, G.: Généralisation des éléments de Delaunay en mécanique céleste. Application au mouvement d'un satellite artificiel, *C.R. Acad. Sc. Paris* 271, 729, 1970.
9. Mueller, A.: The Development of the Poincaré-Similar with the True Anomaly as the Independent Variable, *NASA/JSC Internal Note*, No. 76-FM-60, August, 1976.
10. Mueller, A.: A Comparison of Two Types of Total Energy Elements, *NASA/JSC Internal Note* No. 76-FM-94, December, 1976.
11. Bond, V. and Scheifele, G.: First Order Analytical  $J_2$  Theory in the Poincaré Similar Elements. (To be published).

**PRECEDING PAGE BLANK NOT FILMED**

APPENDIX A

PRECEDING PAGE BLANK NOT FILMED

APPENDIX A

COTOPS

Given  $\underline{x}$ ,  $\dot{\underline{x}}$ , and time, transform to  $\underline{\sigma}$  and  $\underline{\rho}$ .

Evaluate the potential,  $V$ .

Then sequentially compute:

$$\rho_4 = L = \frac{\mu}{r} - \frac{1}{2} (\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - V \quad (A.1)$$

$$G_1 = x_2 \dot{x}_3 - x_3 \dot{x}_2$$

$$G_2 = x_3 \dot{x}_1 - x_1 \dot{x}_3$$

$$G_3 = x_1 \dot{x}_2 - x_2 \dot{x}_1$$

$$G = \sqrt{G_1^2 + G_2^2 + G_3^2}$$

$$H = G_3$$

$$\rho_1 = \phi = G - \sqrt{G^2 - 2r^2 V} + \frac{\mu}{\sqrt{2L}} \quad (A.2)$$

$$q = G - \frac{1}{2} \phi + \frac{\mu}{2\sqrt{2L}}$$

$$p = \frac{1}{\mu} \left( G - \phi + \frac{\mu}{\sqrt{2L}} \right)^2$$

$$Q = \frac{1}{\mu} \left[ \rho_4 \left( \frac{2\mu}{\sqrt{2\rho}} + H \mp G \right) \right]^{\frac{1}{2}}$$

$$\sigma_3 = \frac{-2G_1}{\{2(G+H)\}^{\frac{1}{2}}} \quad (A.3)$$

$$\rho_3 = \frac{-2G_2}{\{2(G+H)\}^{\frac{1}{2}}} \quad (A.4)$$

$$R = \frac{2G\left(\frac{x_3}{r}\right)}{\{2(G+H)\}^{\frac{1}{2}}}$$

$$R^* = \frac{rR}{2G}$$

$$r \cos \sigma_1 = x_1 + R^* \sigma_3$$

$$r \sin \sigma_1 = x_2 + R^* \rho_3$$

$$\sigma_1 = \arctan \left( \frac{r \sin \sigma_1}{r \cos \sigma_1} \right) \quad (A.5)$$

$$\dot{r} = \frac{\underline{x} \cdot \dot{\underline{x}}}{r}$$

$$Z_1 = \frac{\left(\frac{p}{r} - 1\right)}{Q}$$

$$Z_2 = \frac{\dot{r}p}{Q(2q - G)}$$

$$\sigma_2 = Z_2 \cos \sigma_1 - Z_1 \sin \sigma_1 \quad (A.6)$$

$$\rho_2 = Z_2 \sin \sigma_1 + Z_1 \cos \sigma_1 \quad (A.7)$$

$$\sqrt{1-e^2} = \sqrt{\frac{2L}{\mu}} p$$

$$E - \phi = -2 \arctan \left( \frac{Z_2 \cdot Q}{(1 + \sqrt{1-e^2} + Z_1 \cdot Q)} \right)$$

$$\sigma_4 = t - \frac{\mu}{(2\rho_4)^{3/2}} \cdot \left( E - \phi - \frac{r}{p} Z_2 Q \sqrt{1-e^2} \right) \quad (A.8)$$

PSTOCO

Given  $\underline{\sigma}$ ,  $\underline{\rho}$ , transform to  $\underline{x}$ ,  $\underline{x}$ ,  $t$  = time.

Sequentially compute:

$$Z_1 = \rho_2 \cos \sigma_1 - \sigma_2 \sin \sigma_1$$

$$Z_2 = \rho_2 \sin \sigma_1 + \sigma_2 \cos \sigma_1$$

$$Q = \frac{1}{\mu} \left[ \rho_4 \left( \frac{\mu}{\sqrt{2\rho_4}} - \frac{1}{2}(\sigma_2^2 + \rho_2^2) \right) \right]^{\frac{1}{2}}$$



$$e \cos \phi = Z_1 Q$$

$$e \sin \phi = Z_2' Q$$

$$p = \frac{1}{\mu} \left[ -\frac{1}{2} (\rho_2^2 + \sigma_2^2) + \frac{\mu}{\sqrt{2\rho_4}} \right]$$

$$r = \frac{p}{1 + e \cos \phi}$$

$$R = \rho_3 \sin \sigma_1 + \sigma_3 \cos \sigma_1$$

$$G = \rho_1 - \frac{1}{2} (\sigma_2^2 + \rho_2^2)$$

$$H = G - \frac{1}{2} (\rho_3^2 + \sigma_3^2)$$

$$R^* = \frac{rR}{2G}$$

$$x_1 = -R^* \sigma_3 + r \cos \sigma_1 \quad (A.9)$$

$$x_2 = -R^* \rho_3 + r \sin \sigma_1 \quad (A.10)$$

$$x_3 = R^* \sqrt{2(G + H)} \quad (A.11)$$

$$\sqrt{1 - e^2} = \sqrt{\frac{2L}{\mu}} p$$

$$E - \phi = -2 \arctan \left( \frac{e \sin \phi}{1 + \sqrt{1 - e^2} + e \cos \phi} \right)$$

$$t = \sigma_4 + \frac{\mu}{(2\rho_4)^{3/2}} (E - \phi - \frac{r}{p} \sqrt{1 - e^2} e \sin \phi) \quad (A.12)$$

$$q = -\frac{1}{2} \left( \sigma_2^2 + \rho_2^2 - \rho_1 - \frac{\mu}{\sqrt{2\rho_4}} \right)$$

$$\dot{r} = \frac{e \sin \phi}{p} \left[ 2q - \rho_1 + \frac{1}{2} (\sigma_2^2 + \rho_2^2) \right]$$

$$\dot{R}^* = \frac{\dot{r}R + R\dot{r}}{2G}, \quad \dot{R} = \frac{G}{r^2} (\rho_3 \cos \sigma_1 - \sigma_3 \sin \sigma_1)$$

$$\dot{x}_1 = -\dot{R}^* \sigma_3 + \dot{r} \cos \sigma_1 - \frac{G}{r} \sin \sigma_1 \quad (A.13)$$

$$\dot{x}_2 = -\dot{R}^* \rho_3 + \dot{r} \sin \sigma_1 + \frac{G}{r} \cos \sigma_1 \quad (A.14)$$

$$\dot{x}_3 = \dot{R}^* \sqrt{2(G + H)} \quad (A.15)$$

**PRECEDING PAGE BLANK NOT FILMED**

APPENDIX B

PRECEDING PAGE BLANK NOT FILMED

APPENDIX B

If one defines

$$S_{1k} = \frac{\partial S_1}{\partial \sigma_k}$$

then

$$S_{1k} = \frac{1}{G} (w_k y + w y_k - \frac{2wy G_k}{G})$$

where

$$w = \frac{Q}{2pq}$$

$$w_k = \frac{1}{2p^2q^2} [pq Q_k - Q (p_k q + qp_k)]$$

$$y = \sum_{\ell=1}^3 (\delta_{\ell} \eta_{\ell} + \gamma_{\ell} \xi_{\ell})$$

$$y_1 = \sum_{\ell=1}^3 [\ell (\delta_{\ell} \xi_{\ell} - \gamma_{\ell} \eta_{\ell}) + \delta_{\ell 1} \eta_{\ell} + \gamma_{\ell} \xi_{\ell}]$$

$$y_k = \sum_{\ell=1}^3 (\delta_{\ell k} \eta_{\ell} + \gamma_{\ell k} \xi_{\ell}) \quad k=2,3,\dots,8$$

$$G = \sigma_5 - \frac{1}{2} (\sigma_2^2 + \sigma_6^2)$$

$$G_k = 0 \quad \text{for } k=1,3,4,7,8$$

$$G_2 = -\sigma_2$$

$$G_5 = 1$$

$$G_6 = -\sigma_6$$

Here  $p, p_k, q, q_k, Q, Q_k, \delta_\ell, \eta_\ell, \gamma_\ell, \xi_\ell$  and  $\delta_{\ell k}, \gamma_{\ell k}$  are 'displayed'.

$$p = \frac{1}{\mu} \left[ -\frac{1}{2} (\sigma_2^2 + \sigma_6^2) + \frac{\mu}{\sqrt{2\sigma_8}} \right]^2$$

$$p_2 = -2 \frac{\sqrt{\mu p}}{\mu} \sigma_2$$

$$p_6 = -2 \frac{\sqrt{\mu p}}{\mu} \sigma_6$$

$$p_8 = -2 \frac{\sqrt{\mu p}}{(2\sigma_8)^{3/2}}$$

$$p_k = 0 \quad \text{for } k=1, 3, 4, 5, 7$$

$$q = -\frac{1}{2} (\sigma_6^2 + \sigma_2^2 - \sigma_5) + \frac{\mu}{2\sqrt{2\sigma_8}}$$

$$q_2 = -\sigma_2$$

$$q_5 = \frac{1}{2}$$

$$q_8 = -\frac{\mu}{2} \frac{1}{(2\sigma_8)^{3/2}}$$

$$Q = \left[ \frac{\sigma_8}{\mu^2} \left( \frac{2\mu}{\sqrt{2\sigma_8}} - \frac{1}{2} (\sigma_6^2 + \sigma_2^2) \right) \right]^{\frac{1}{2}}$$

$$Q_2 = -\frac{\sigma_8 \sigma_2}{2Q\mu^2}$$

$$Q_6 = -\frac{\sigma_8 \sigma_6}{2Q\mu^2}$$

$$Q_8 = \frac{\sqrt{\mu p}}{2Q\mu^2}$$

$$Q_k = 0 \quad \text{for } k=1,3,4,5,7$$

$$\delta_1 = \frac{B}{3} \sigma_6 - \frac{1}{2} (\sigma_6 c - \sigma_2 s)$$

$$\delta_{12} = \frac{s}{2} + \frac{\sigma_6}{3} B_2 - \frac{1}{2} (\sigma_6 c_2 - \sigma_2 s_2)$$

$$\delta_{16} = \frac{B}{3} - \frac{c}{2} + \frac{\sigma_6}{3} B_6 - \frac{1}{2} (\sigma_6 c_6 - \sigma_2 s_6)$$

$$\delta_{1k} = \frac{\sigma_6}{3} B_k - \frac{1}{2} (\sigma_6 c_k - \sigma_2 s_k) \quad \text{for } k=1,3,4,5,7,8$$

$$\gamma_1 = \frac{B}{3} \sigma_2 + \frac{1}{2} (\sigma_6 s + \sigma_2 c)$$

$$\gamma_{12} = \left( \frac{B}{3} + \frac{c}{2} \right) + \frac{\sigma_2}{3} B_2 + \frac{1}{2} (\sigma_6 s_2 + \sigma_2 c_2)$$

$$\gamma_{16} = \frac{s}{2} + \frac{\sigma_2}{3} B_6 + \frac{1}{2} (\sigma_6 s_6 + \sigma_2 c_6)$$

$$\gamma_{1k} = \frac{\sigma_2}{3} B_k + \frac{1}{2} (\sigma_6 s_k + \sigma_2 c_k) \quad k=1,3,4,5,7,8$$

$$\delta_2 = - \frac{c}{2Q}$$

$$\delta_{2k} = \frac{1}{2Q} \left[ \frac{c}{Q} Q_k - c_k \right]$$

$$\gamma_2 = \frac{s}{2Q}$$

$$\gamma_{2k} = - \frac{1}{2Q} \left( \frac{s}{Q} Q_k - s_k \right)$$

$$\delta_3 = - \frac{1}{6} (\sigma_2 s + \sigma_6 c)$$

$$\delta_{32} = - \frac{1}{6} (\sigma_2 s_2 + \sigma_2 c_2 + s)$$

$$\delta_{36} = - \frac{1}{6} (\sigma_2 s_6 + \sigma_6 c_6 + c)$$

$$\delta_{3k} = - \frac{1}{6} (\sigma_2 s_k + \sigma_6 c_k) \quad k=1,3,4,5,7,8$$

$$\gamma_3 = \frac{1}{6} (\sigma_6 s - \sigma_2 c)$$

$$\gamma_{32} = \frac{1}{6} (\sigma_6 s_2 - \sigma_2 c_2 - c)$$

$$\gamma_{36} = \frac{1}{6} (\sigma_6 s_6 - \sigma_2 c_6 + s)$$

$$\gamma_{3k} = \frac{1}{6} (\sigma_6 s_k - \sigma_2 c_k) \quad k=1,3,4,5,7,8$$

$$\eta_{\ell} = \sin \ell \sigma_1$$

$$\xi_{\ell} = \cos \ell \sigma_1$$

Here  $c$ ,  $s$ ,  $c_k$ ,  $s_k$ ,  $B$ ,  $B_k$ ,  $H$  and  $H_k$  are displayed

$$c = (G+H) \left( \frac{\sigma_7^2 - \sigma_3^2}{2} \right)$$

$$c_3 = \frac{H_3 c}{(G+H)} - (G+H) \sigma_3$$

$$c_7 = \frac{H_7 c}{(G+H)} + (G+H) \sigma_7$$

$$c_k = \frac{G_k + H_k}{(G+H)} c \quad \text{for } k=1,2,4,5,6,8$$

$$s = - (G+H) \sigma_3 \sigma_7$$

$$s_3 = \frac{H_3 s}{G+H} - (G+H) \sigma_7$$

$$s_7 = \frac{H_7 s}{G+H} - (G+H) \sigma_3$$

$$s_k = \frac{(G_k + H_k)}{(G+H)} s \quad \text{for } k=1,2,4,5,6,8$$



$$B = G^2 - 3 H^2$$

$$B_k = 2 (G G_k - 3H H_k)$$

$$H = G - \frac{1}{2} (\sigma_3^2 + \sigma_7^2)$$

$$H_3 = - \sigma_3$$

$$H_7 = - \sigma_7$$

$$H_k = G_k \quad \text{for } k=1,2,4,5,6,8$$

Abbreviations used in the integration of the primed system

$$A_4 = 1 + \frac{\varepsilon}{2} f_4(b - \frac{2}{3})$$

$$A_3 = \frac{\varepsilon}{2} f b_2$$

$$A_2 = \frac{\varepsilon}{2} \left[ f_2(b - \frac{2}{3}) + f b_2 \right] + A_3$$

$$A_1 = \frac{\varepsilon}{2} f_1(b - \frac{2}{3}) + A_2$$

$$f = \frac{1}{pq}$$

$$f_2 = - \frac{f^2}{\mu} (\mu p + 2q \sqrt{\mu p})$$

$$f_4 = \frac{f^2}{(2\rho_4)^{3/2}} \left( \frac{1}{2} \mu p + 2q \sqrt{\mu p} \right)$$

$$b = 1 - \frac{H^2}{G^2}$$

$$b_2 = \frac{2}{G} \left( \frac{H}{G} \right)^2$$

$$b_3 = - \frac{2}{G} \left( \frac{H}{G} \right)$$

**PRECEDING PAGE BLANK NOT FILMED**

APPENDIX C

PRECEDING PAGE BLANK NOT FILMED  
APPENDIX C

If it is defined

$$S_{1kj} = \frac{\partial(S_1)}{\partial\sigma_k\sigma_j}$$

then the expression for  $S_{1kj}$  is as follows

$$S_{1kj} = -\frac{2}{G} (S_{1k} G_j + S_{1j} G_k) - \frac{1}{G^2} \left[ w_k y_j + w_{kj} y + w_j y_k + w_{jk} y - \frac{2wy}{G} (G_{kj} + \frac{G_j G_k}{G}) \right]$$

where  $S_{1k}$ ,  $G_j$ ,  $w_k$ ,  $y_k$ ,  $y$ ,  $G$ , and  $w$  may be found in Appendix B.

The expressions for  $y_{kj}$ ,  $w_{kj}$  and  $G_{kj}$  are:

$$y_{kj} = \sum_{\ell=1}^3 (\delta_{\ell kj} \eta_{\ell} + \gamma_{\ell kj} \xi_{\ell}) \quad \begin{matrix} k = 2, 3, \dots, 8 \\ j = 2, 3, \dots, 8 \end{matrix}$$

$$y_{1j} = \sum_{\ell=1}^3 \ell (\delta_{\ell j} \xi_{\ell} - \gamma_{\ell j} \eta_{\ell}) \quad j = 2, 3, \dots, 8$$

$$y_{11} = \sum_{\ell=1}^3 \ell (\delta_{\ell 1} \xi_{\ell} - \gamma_{\ell 1} \eta_{\ell}) - \ell^2 (\delta_{\ell} \eta_{\ell} + \gamma_{\ell} \xi_{\ell})$$

$$w_{kj} = \left( \frac{Q_k}{2p^2 q^2} - \frac{2w_k}{pq} \right) (p_j q + q_j p) + \frac{1}{2p^2 q^2} \left[ pq Q_{kj} - Q_j (pq_k + qp_k) - Q (p_j q_k + pq_{kj} + q_j p_k + qp_{kj}) \right]$$

$$G_{jk} = 0 \quad \begin{matrix} j = 1, 2, & 8 \\ k = 1, 2, & 8 \end{matrix} \quad \begin{matrix} \text{except } j = k = 6 \\ j = k = 2 \end{matrix}$$

$$G_{22} = 1, \quad G_{66} = -1$$

where  $\eta_\ell, \xi_\ell, \delta_{\ell k}, \gamma_{\ell j}, \delta_\ell, \gamma_\ell, p, q, p_k, q_k, Q, Q_k$  are found in Appendix B.

Expressions for  $\delta_{\ell kj}$  and  $\gamma_{\ell kj}$  are as follows:

$$\delta_{3jk} = \delta_{3kj}$$

$$\delta_{3kj} = -\frac{1}{6}(\sigma_2 s_{kj} + \sigma_6 c_{kj}) \quad \begin{matrix} \text{for } k = 1, 3, 4, 5, 7, 8 \\ j = 1, 2, \dots, k \end{matrix}$$

$$\delta_{32j} = -\frac{1}{6}(\sigma_2 s_{2j} + \sigma_6 c_{2j} + s_j) \quad j = 1, 2$$

$$\delta_{36j} = -\frac{1}{6}(\sigma_2 s_{6j} + \sigma_6 c_{6j} + c_j) \quad j = 1, 2, \dots, 6$$

$$\gamma_{3jk} = \gamma_{3kj}$$

$$\gamma_{3kj} = \frac{1}{6}(\sigma_6 s_{kj} - \sigma_2 c_{kj}) \quad \begin{matrix} k = 1, 3, 4, 5, 7, 8 \\ j = 1, 2, \dots, k \end{matrix}$$

$$\gamma_{32j} = \frac{1}{6}(\sigma_6 s_{2j} - \sigma_2 c_{2j} - c_j) \quad j = 1, 2$$

$$\gamma_{36j} = \frac{1}{6}(\sigma_6 s_{6j} - \sigma_2 c_{6j} + s_j) \quad j = 1, 2, \dots, 6$$

$$\delta_{2kj} = -\frac{1}{Q} \left[ \delta_{2k} Q_j + \delta_{2j} Q_k - \frac{1}{2} \left( \frac{c}{Q} Q_{kj} - c_{kj} \right) \right] \quad \begin{matrix} k = 1, 2, \dots, 8 \\ j = 1, 2, \dots, 8 \end{matrix}$$

$$\gamma_{2kj} = -\frac{1}{Q} \left[ \gamma_{2k} Q_j + \gamma_{2j} Q_k + \frac{1}{2} \left( \frac{s}{Q} Q_{kj} - s_{kj} \right) \right]$$

$$\delta_{1jk} = \delta_{1kj}$$

$$\delta_{1kj} = \frac{\sigma_6}{3} B_{kj} - \frac{1}{2} (\sigma_6 c_{kj} - \sigma_2 s_{kj}) \quad \begin{matrix} k = 1, 3, 4, 5, 7, 8 \\ j = 1, 2, \dots, k \end{matrix}$$

$$\delta_{12j} = \frac{\sigma_6}{3} B_{2j} - \frac{1}{2} (\sigma_6 c_{2j} - \sigma_2 s_{2j} - s_j) \quad j = 1, 2$$

$$\delta_{16j} = \frac{1}{3} (\sigma_6 B_{6j} + B_j) - \frac{1}{2} (\sigma_6 c_{6j} - \sigma_2 s_{6j} + c_j) \quad j = 1, 2, \dots, 6$$

$$\gamma_{1jk} = \gamma_{1kj}$$

$$\gamma_{1kj} = \frac{\sigma_2}{3} B_{kj} + \frac{1}{2} (\sigma_6 s_{kj} + \sigma_2 c_{kj}) \quad \begin{matrix} k = 1, 3, 4, 5, 7, 8 \\ j = 1, 2, \dots, k \end{matrix}$$

$$\gamma_{12j} = \frac{1}{3} (\sigma_2 B_{2j} + B_j) + \frac{1}{2} (\sigma_6 s_{2j} + \sigma_2 c_{2j} + c_j) \quad j = 1, 2$$

$$\gamma_{16j} = \frac{\sigma_2}{3} B_{6j} + \frac{1}{2} (\sigma_6 s_{6j} + \sigma_2 c_{6j} + s_j) \quad j = 1, 2, \dots, 6$$

where  $B$ ,  $B_j$ ,  $c$ ,  $c_j$ ,  $s$ , and  $s_j$  may be found in Appendix B.

The expression for  $s_{kj}$ ,  $c_{kj}$ ,  $B_{kj}$ ,  $q_{kj}$ ,  $p_{kj}$ , and  $Q_{kj}$  are listed here:

$$s_{jk} = s_{kj}$$

$$s_{1j} = s_{4j} = s_{8j} = 0 \quad j = 1, 2, \dots, 8$$

$$s_{22} = 2 \sigma_7 \sigma_3$$

$$s_{23} = 2 \sigma_7 \sigma_2$$

$$s_{25} = s_{26} = 0$$

$$s_{27} = 2 \sigma_2 \sigma_3$$

$$s_{33} = - \sigma_7 (G_3 + H_3 - 2 \sigma_3)$$

$$s_{35} = - \sigma_7 (G_5 + H_5)$$

$$s_{36} = - \sigma_7 (G_6 + H_6)$$

$$s_{37} = - (G + H - \sigma_3^2) - \sigma_7 (G_7 + H_7)$$

$$s_{55} = s_{56} = 0$$

$$s_{57} = - 2 \sigma_3$$

$$s_{66} = 2 \sigma_3 \sigma_7$$

$$s_{67} = 2 \sigma_3 \sigma_6$$

$$s_{77} = - \sigma_3 (G_7 + H_7 - 2 \sigma_7)$$

$$c_{jk} = c_{kj}$$

$$c_{kj} = 0 \quad k = 1, 4, 8$$

$$c_{22} = \sigma_3^2 - \sigma_7^2$$

$$c_{23} = 2 \sigma_3 \sigma_2$$

$$c_{25} = 0$$

$$c_{26} = 0$$

$$c_{27} = - 2 \sigma_2 \sigma_7$$

$$c_{33} = - \left( \frac{\sigma_7^2 - \sigma_3^2}{2} + G + H \right) - \sigma_3 (G_3 + H_3 - \sigma_3)$$

$$c_{35} = - \sigma_3 (G_5 + H_5)$$

$$c_{36} = - \sigma_3 (G_6 + H_6)$$

$$c_{37} = - \sigma_3 (\sigma_7 + G_7 + H_7)$$

$$c_{55} = c_{56} = 0$$



$$c_{57} = 2 \sigma_7$$

$$c_{66} = \sigma_3^2 - \sigma_7^2$$

$$c_{67} = - 2 \sigma_6 \sigma_7$$

$$c_{77} = \left( \frac{\sigma_3^2 - \sigma_7^2}{2} + G + H \right) + \sigma_7 (G_7 + H_7 - \sigma_7)$$

$$B_{jk} = B_{kj}$$

$$B_{kj} = 0 \quad \text{for } k = 1, 4, 8$$

$$B_{2j} = - 2\sigma_2 (G_j - 3H_j) \quad j = 3, 4, \dots, 7$$

$$B_{22} = - 2 \sigma_2 (G_2 - 3H_2) - 2 (G - 3H)$$

$$B_{3j} = 6 \sigma_3 H_j \quad j = 4, 5, \dots, 8$$

$$B_{33} = 6 \sigma_3 H_3 + 6H$$

$$B_{5j} = 2(G_j - 3H_j) \quad j = 5, 6, 7, 8$$

$$B_{6j} = - 2\sigma_6 (G_j - 3H_j) \quad j = 7, 8$$

$$B_{66} = - 2 \sigma_6 (G_6 - 3H_6) - 2 (G - 3H)$$

$$B_{77} = 6 \sigma_7 H_7 + 6H$$

$$B_{78} = 6 \sigma_7 H_7$$

The expressions for  $H_j$  and  $G_j$  may be found in Appendix B.

$$q_{kj} = 0 \text{ for all } k \text{ \& } j \text{ except ,}$$

$$q_{22} = - 1$$

$$q_{66} = - 1$$

$$q_{88} = - \frac{3q_8}{2\sigma_8}$$

$$p_{kj} = 0 \text{ for all } k \text{ \& } j \text{ except ,}$$

$$p_{22} = - 2 \frac{(\mu p)^{\frac{1}{2}}}{\mu} + \frac{p_2^2}{2p}$$

$$p_{26} = p_{62} = \frac{p_2 p_6}{2p}$$

$$p_{28} = p_{82} = \frac{p_2 p_8}{2p}$$

$$p_{66} = - 2 \frac{(\mu p)^{\frac{1}{2}}}{\mu} + \frac{p_6^2}{2p}$$

$$p_{68} = p_{86} = \frac{p_6 p_8}{2p}$$

$$p_{88} = -\frac{3p_8}{2\sigma_8} + \frac{p_8^2}{2p}$$

$$Q_{kj} = 0 \quad \text{for all } k \text{ \& } j \text{ except}$$

$$Q_{22} = -\frac{\sigma_8}{2Q\mu^2} - \frac{Q_2^2}{Q}$$

$$Q_{26} = Q_{62} = -\frac{Q_2 Q_6}{Q}$$

$$Q_{28} = Q_{82} = -\frac{\sigma_8}{2Q\mu^2} - \frac{Q_6^2}{Q}$$

$$Q_{68} = Q_{86} = -\frac{\sigma_6}{2Q\mu^2} - \frac{Q_6 Q_8}{Q}$$

$$Q_{66} = -\frac{\sigma_8}{2Q\mu^2} - \frac{Q_6^2}{Q}$$

$$Q_{88} = \frac{Q_8 p_8}{2p} - \frac{Q_8^2}{Q}$$

$A_{kj}$  is defined as follows

$$A_{kj} = \frac{\partial A_k}{\partial \sigma_j}$$

Expressions for  $A_{kj}$  are:

$$A_{4j} = \frac{\varepsilon}{2} \left[ f_{4j} \left( b - \frac{2}{3} \right) + f_4 \left( \tilde{b}_j - \frac{2}{3} \right) \right] \quad j = 1, 2, \dots, 7$$

$$A_{48} = \frac{\varepsilon}{2} \left[ f_{48} \left( b - \frac{2}{3} \right) + f_4 \left( \tilde{b}_8 - \frac{2}{3} \right) \right] - \frac{3\mu}{(2\sigma_8)^{5/2}}$$

$$\left. \begin{aligned} A_{3j} &= \frac{\varepsilon}{2} \left[ \tilde{f}_j b_3 + f b_{3j} \right] \\ A_{2j} &= \frac{\varepsilon}{2} \left[ f_{2j} \left( b - \frac{2}{3} \right) + f_2 \tilde{b}_j + \tilde{f}_j b_2 + f b_{2j} \right] + A_{3j} \\ A_{1j} &= \frac{\varepsilon}{2} \left[ f_{1j} \left( b - \frac{2}{3} \right) + f_1 \tilde{b}_j \right] + A_{2j} \end{aligned} \right\} \quad j = 1, 2, \dots, 8$$

where

$$\tilde{f}_j = \frac{\partial f}{\partial \sigma_j}, \quad \tilde{b}_j = \frac{\partial b}{\partial \sigma_j}$$

$$f_{1j} = \frac{\partial f_1}{\partial \sigma_j}, \quad f_{2j} = \frac{\partial f_2}{\partial \sigma_j}, \quad f_{4j} = \frac{\partial f_4}{\partial \sigma_j}$$

$$b_{2j} = \frac{\partial b_2}{\partial \sigma_j}, \quad b_{3j} = \frac{\partial b_3}{\partial \sigma_j}$$

Expressions for  $f$ ,  $f_1$ ,  $f_2$ ,  $f_4$ ,  $b$ ,  $b_2$  and  $b_3$  may be found in Appendix B.

If the abbreviations are made

$$z = \frac{1}{2} \mu p + 2 q (\mu p)^{\frac{1}{2}}$$

$$z_j = \mu p_j \left( \frac{1}{2} + \frac{q}{(\mu p)^{\frac{1}{2}}} \right) + 2 q_j (\mu p)^{\frac{1}{2}}$$

then the expressions for  $\tilde{f}_j$ ,  $\tilde{b}_j$ ,  $f_{1j}$ ,  $f_{2j}$ ,  $f_{4j}$ ,  $b_{2j}$ ,  $b_{3j}$  are:

$$\tilde{f}_j = -f^2 (p q_j + q p_j)$$

$$f_{1j} = \frac{2\tilde{f}_1 \tilde{f}_j}{f} + \frac{f^2 z_j}{\mu}$$

$$f_{2j} = \frac{2\tilde{f}_2 \tilde{f}_j}{f} - \frac{f^2}{\mu} \left( \frac{1}{2} \mu p_j + z_j \right)$$

$$f_{4j} = \frac{2\tilde{f}_4 \tilde{f}_j}{f} + \frac{f^2}{(2\sigma_8)^{3/2}} z_j \quad j = 1, 2, \dots, 7$$

$$f_{48} = \frac{2\tilde{f}_4 \tilde{f}_8}{f} + \frac{f^2}{(2\sigma_8)^{3/2}} z_8 - \frac{3f_4}{2\sigma_8}$$

$$\tilde{b}_j = \frac{2H}{G^2} \left( \frac{H}{G} G_j - H_j \right)$$

$$b_{2j} = \frac{2H}{G^3} \left( 2 H_j - \frac{3H}{G} G_j \right)$$

$$b_{3j} = \frac{2}{G^2} (2 \frac{H}{G} G_j - H_j)$$

Expressions for  $\frac{d\sigma'_k}{d\tau}$  are:

$$\frac{d\sigma'_1}{d\tau} = A_1$$

$$\frac{d\sigma'_2}{d\tau} = -\sigma'_6 A_2$$

$$\frac{d\sigma'_3}{d\tau} = -\sigma'_7 A_3$$

$$\frac{d\sigma'_4}{d\tau} = A_4$$

$$\frac{d\sigma'_5}{d\tau} = 0$$

$$\frac{d\sigma'_6}{d\tau} = \sigma'_2 A_2$$

$$\frac{d\sigma'_7}{d\tau} = \sigma'_3 A_3$$

$$\frac{d\sigma'_8}{d\tau} = 0$$

PRECEDING PAGE BLANK NOT FILMED

APPENDIX D

PRECEDING PAGE BLANK NOT FILMED

# APPENDIX D

If one defines

$$\frac{\partial y_k}{\partial \sigma_j} = y_{kj}$$

then

$$y_{11} = \frac{x_1}{r} \frac{\partial r}{\partial \sigma_1} - r \sin \sigma_1 - \frac{\sigma_3 r}{2G} (\rho_3 \cos \sigma_1 - \sigma_3 \sin \sigma_1)$$

$$y_{12} = \frac{x_1}{r} \frac{\partial r}{\partial \sigma_2} - \sigma_3 \sigma_2 \frac{R^*}{G}$$

$$y_{13} = - R^* - \frac{\sigma_3 r}{2G} \cos \sigma_1$$

$$y_{14} = 0$$

$$y_{21} = \frac{x_2}{r} \frac{\partial r}{\partial \sigma_1} + r \cos \sigma_1 - \frac{r \rho_3}{2G} (\rho_3 \cos \sigma_1 - \sigma_3 \sin \sigma_1)$$

$$y_{22} = \frac{x_2}{r} \frac{\partial r}{\partial \sigma_2} - \sigma_2 \cdot y_{21}$$



$$y_{23} = \frac{-r}{2G} \rho_3 \cos \sigma_1$$

$$y_{24} = 0$$

$$y_{31} = \frac{r}{2G} \sqrt{2(G+H)} (\rho_3 \cos \sigma_1 - \sigma_3 \sin \sigma_1) + \frac{x_3}{r} \frac{\partial r}{\partial \sigma_1}$$

$$y_{32} = -\sigma_2 y_{31} + \frac{x_3}{r} \frac{\partial r}{\partial \sigma_2}$$

$$y_{33} = \frac{-\sigma_3}{2} y_{31} + \frac{x_1}{2G} \sqrt{2(G+H)}$$

$$y_{34} = 0$$

$$y_{4k} = \frac{\mu}{(2\rho_4)^{3/2}} \left[ \frac{\partial(E-\phi)}{\partial \sigma_k} - \frac{r}{p} \left\{ \sqrt{1-e^2} \frac{\partial(e \sin \phi)}{\partial \sigma_k} \right. \right. \\ \left. \left. + e \sin \phi \frac{\partial \sqrt{1-e^2}}{\partial \sigma_k} + \frac{r}{p} \frac{\partial(e \cos \phi)}{\partial \sigma_k} \sqrt{1-e^2} e \sin \phi \right\} \right]$$

$$k=1, 2, 3$$

$$y_{44} = 1$$

$$y_{15} = \frac{\sigma_3 R^*}{G}$$

$$y_{16} = \frac{x_1}{r} \frac{\partial r}{\partial \rho_2} - \sigma_3 \rho_2 \frac{R^*}{G}$$

$$y_{17} = \frac{-\sigma_3 r}{2G} \sin \sigma_1$$

$$y_{18} = \frac{x_1}{r} \frac{\partial r}{\partial \rho_4}$$

$$y_{25} = \frac{\rho_3 R^*}{G}$$

$$y_{26} = \frac{x_2}{r} \frac{\partial r}{\partial \rho_2} - \frac{R^*}{G} \rho_3 \rho_2$$

$$y_{27} = -R^* - \frac{\rho_3 r}{2G} \sin \sigma_1$$

$$y_{28} = \frac{x_2}{r} \frac{\partial r}{\partial \rho_4}$$

$$y_{35} = -x_3 \frac{H}{G(G + H)}$$

$$y_{36} = \rho_2 x_3 \frac{H}{G(G + H)} + \frac{x_3}{r} \frac{\partial r}{\partial \rho_2}$$

$$y_{37} = \frac{\rho_3 x_3}{2G(G + H)} + x_2 \frac{\sqrt{2(G + H)}}{2G}$$

$$y_{38} = \frac{x_3}{r} \frac{\partial r}{\partial \rho_4}$$

$$y_{4,k} = \frac{\mu}{(2\rho_4)^{3/2}} \left[ \frac{\partial(E - \phi)}{\partial \rho_k} - \frac{r}{p} \left\{ \sqrt{1 - e^2} \frac{\partial(e \sin \phi)}{\partial \rho_k} \right. \right. \\ \left. \left. + e \sin \phi \frac{\partial \sqrt{1 - e^2}}{\partial \rho_k} + \frac{r}{p} \frac{\partial(e \cos \phi)}{\partial \rho_k} \sqrt{1 - e^2} e \sin \phi \right\} \right]$$

$$k=5,6,7,8$$

$$y_{48} = y_{48} - \frac{3}{2} \left( \frac{x_4 - \sigma_4}{\rho_4} \right)$$

for k = 1,2,3,4:

$$y_{5k} = -f \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \sigma_3 \frac{\partial f}{\partial \sigma_k} + \cos \sigma_1 \frac{\partial \dot{r}}{\partial \sigma_k} - \dot{r} \begin{pmatrix} \sin \sigma_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$+ \frac{G}{r^2} \sin \sigma_1 \frac{\partial r}{\partial \sigma_k} - \frac{\sin \sigma_1}{r} \frac{\partial G}{\partial \sigma_k} - \frac{G}{r} \begin{pmatrix} \cos \sigma_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$y_{6k} = -\rho_3 \frac{\partial f}{\partial \sigma_k} + \sin \sigma_1 \frac{\partial \dot{r}}{\partial \sigma_k} + \dot{r} \begin{pmatrix} \cos \sigma_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$- \frac{G}{r^2} \cos \sigma_1 \frac{\partial r}{\partial \sigma_k} + \frac{\cos \sigma_1}{r} \frac{\partial G}{\partial \sigma_k} - \frac{G}{r} \begin{pmatrix} \sin \sigma_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$y_{7k} = \frac{\dot{y}_7}{2(G+H)^{\frac{1}{2}}} \frac{\partial(G+H)}{\partial\sigma_k} + \left[2(G+H)\right]^{\frac{1}{2}} \frac{\partial f}{\partial\sigma_k}$$

for k = 5, 6, 7, 8:

$$y_{5k} = -\sigma_3 \frac{\partial f}{\partial\sigma_k} + \cos \sigma_1 \frac{\partial \dot{r}}{\partial\sigma_k} + \frac{G}{r^2} \sin \sigma_1 \frac{\partial r}{\partial\sigma_k} \\ - \frac{\sin \sigma_1}{r} \frac{\partial G}{\partial\sigma_k}$$

$$y_{6k} = -f \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \rho_3 \frac{\partial f}{\partial\sigma_k} + \sin \sigma_1 \frac{\partial r}{\partial\sigma_k} - \frac{G}{r^2} \cos \sigma_1 \frac{\partial r}{\partial\sigma_k} \\ + \frac{\cos \sigma_1}{r} \frac{\partial G}{\partial\sigma_k}$$

$$y_{7k} = \frac{y}{2(G+H)} \frac{\partial(G+H)}{\partial\sigma_k} + \left[2(G+H)\right]^{\frac{1}{2}} \frac{\partial f}{\partial\sigma_k}$$

The derivatives of the energy are

$$\frac{\partial y_8}{\partial\sigma_8} = 1 \quad \frac{\partial y_8}{\partial\sigma_k} = 0, \quad (k=1, 2, \dots, 7)$$

Other abbreviations required are:

$$\frac{\partial r}{\partial \rho} = 0 \quad k=1 \text{ and } 3$$

$$\frac{\partial r}{\partial \rho_2} = \frac{r}{p} \left[ -2 \rho_2 \sqrt{\frac{p}{\mu}} - r \left( Q \cos \sigma_1 - \frac{Z_1 \rho_4}{2\mu^2 Q} \rho_2 \right) \right]$$

$$\frac{\partial r}{\partial \rho_4} = \frac{r}{p} \left[ \frac{-2\mu}{(2\rho_4)^{3/2}} \sqrt{\frac{p}{\mu}} - r Z_1 \left( \frac{Q}{2\rho_4} - \frac{Z_1}{2\mu Q (2\rho_4)^{1/2}} \right) \right]$$

$$\frac{\partial r}{\partial \sigma_k} = 0 \quad k=3 \text{ and } 4$$

$$\frac{\partial r}{\partial \sigma_1} = \frac{r^2}{p} Z_2 Q$$

$$\frac{\partial r}{\partial \sigma_2} = \frac{r}{p} \left[ -2 \sigma_2 \sqrt{\frac{p}{\mu}} - r \left( Q \cos \sigma_1 - \frac{Z_1 \rho_4 \sigma_2}{2\mu^2 Q} \right) \right]$$

$$\begin{aligned} \frac{\partial (E - \phi)}{\partial \sigma_k} &= \frac{-r/p}{(1 + \sqrt{1 - e^2})} \left\{ (1 + \sqrt{1 - e^2} + e \cos \phi) \frac{\partial (e \sin \phi)}{\partial \sigma_k} \right. \\ &\quad \left. - e \sin \phi \left[ \frac{\partial \sqrt{1 - e^2}}{\partial \sigma_k} + \frac{\partial (e \cos \phi)}{\partial \sigma_k} \right] \right\} \end{aligned}$$

$$\frac{\partial(E - \phi)}{\partial \rho_k} = \frac{-r/p}{(1 + \sqrt{1 - e^2})} \left\{ (1 + \sqrt{1 - e^2} + e \cos \phi) \frac{\partial(e \sin \phi)}{\partial \rho_k} \right. \\ \left. - e \sin \phi \left[ \frac{\partial \sqrt{1 - e^2}}{\partial \rho_k} + \frac{(e \cos \phi)}{\partial \rho_k} \right] \right\}$$

$$\frac{\partial(e \sin \phi)}{\partial \rho_k} = 0, \quad k=1 \text{ and } 3$$

$$\frac{\partial(e \sin \phi)}{\partial \rho_2} = Q \sin \sigma_1 - \frac{Z_2 \rho_4 \rho_2}{2\mu^2 Q}$$

$$\frac{\partial(e \sin \phi)}{\partial \rho_4} = Z_2 \left[ \frac{Q}{2\rho_4} - \frac{1}{2\mu Q (2\rho_4)^{\frac{1}{2}}} \right]$$

$$\frac{\partial(e \sin \phi)}{\partial \sigma_k} = 0, \quad k=3 \text{ and } 4$$

$$\frac{\partial(e \sin \phi)}{\partial \sigma_1} = e \cos \phi$$

$$\frac{\partial(e \sin \phi)}{\partial \sigma_2} = Q \cos \sigma_1 - Z_2 \frac{\rho_4 \sigma_2}{2\mu^2 Q}$$

$$\frac{\partial(e \cos \phi)}{\partial \rho_k} = 0, \quad k=1 \text{ and } 3$$

$$\frac{\partial(e \cos \phi)}{\partial \rho_2} = Q \cos \sigma_1 - \frac{Z_1 \rho_4 \rho_2}{2\mu^2 Q}$$

$$\frac{\partial(e \cos \phi)}{\partial \rho_4} = Z_1 \left[ \frac{Q}{2\rho_4} - \frac{1}{2\mu Q (2\rho_4)^{\frac{1}{2}}} \right]$$

$$\frac{\partial(e \cos \phi)}{\partial \sigma_k} = 0, \quad k=3 \text{ and } 4$$

$$\frac{\partial(e \cos \phi)}{\partial \sigma_1} = -e \sin \phi$$

$$\frac{\partial(e \cos \phi)}{\partial \sigma_2} = - \left( Q \sin \sigma_1 + \frac{Z_1 \rho_4 \sigma_2}{2\mu^2 Q} \right)$$

$$\frac{\partial \sqrt{1 - e^2}}{\partial \sigma_k} = 0, \quad k=1, 3 \text{ and } 4$$

$$\frac{\partial \sqrt{1 - e^2}}{\partial \sigma_2} = \frac{-\sigma_2}{\mu} \sqrt{2\rho_4}$$

$$\frac{\partial \sqrt{1 - e^2}}{\partial \rho_k} = 0, \quad k=1 \text{ and } 3$$

$$\frac{\partial \sqrt{1 - e^2}}{\partial \rho_2} = \frac{-\rho_2}{\mu} \sqrt{2\rho_4}$$

$$\frac{\partial \sqrt{1 - e^2}}{\partial \rho_4} = \frac{\sqrt{p}}{2\mu\rho_4} - \frac{1}{2\rho_4}$$

Other abbreviations required are:

$$\frac{\partial \dot{r}}{\partial \sigma_1} = \frac{\partial(e \sin \phi)}{\partial \sigma_1} \left( \frac{2q - G}{p} \right)$$

$$\frac{\partial \dot{r}}{\partial \sigma_2} = \frac{\partial(e \sin \phi)}{\partial \sigma_2} \left( \frac{2q - G}{p} \right)$$

$$\frac{\partial \dot{r}}{\partial \sigma_6} = \frac{\partial(e \sin \phi)}{\partial \sigma_6} \left( \frac{2q - G}{p} \right) + \sigma_6 \left[ 2\dot{r} \sqrt{\frac{p}{\mu}} - e \sin \phi \right]$$

$$\frac{\partial \dot{r}}{\partial \sigma_8} = \frac{\partial(e \sin \phi)}{\partial \sigma_8} \left( \frac{2q - G}{p} \right) + \frac{\mu}{(2\sigma_8)^{3/2}} \left( \frac{2\dot{r}}{\mu\sqrt{p}} - \frac{e \sin \phi}{p} \right)$$

$$\frac{\partial \dot{r}}{\partial \sigma_3} = \frac{\partial \dot{r}}{\partial \sigma_4} = \frac{\partial r}{\partial \sigma_5} = \frac{\partial r}{\partial \sigma_7} = 0$$

$$\frac{\partial \dot{R}}{\partial \sigma_1} = -\frac{G}{r^2} R - 2 \frac{\dot{R}}{r} \frac{\partial r}{\partial \sigma_1}$$



$$\frac{\partial \dot{R}}{\partial \sigma_2} = - \frac{\dot{R}}{G} \sigma_2 - \frac{2\dot{R}}{r} \frac{\partial r}{\partial \sigma_2}$$

$$\frac{\partial \dot{R}}{\partial \sigma_3} = - \frac{G}{r^2} \sin \sigma_1$$

$$\frac{\partial \dot{R}}{\partial \sigma_4} = 0$$

$$\frac{\partial \dot{R}}{\partial \sigma_5} = \frac{\dot{R}}{G}$$

$$\frac{\partial \dot{R}}{\partial \sigma_6} = - \frac{\dot{R}}{G} \sigma_6 - \frac{2\dot{R}}{r} \frac{\partial r}{\partial \sigma_6}$$

$$\frac{\partial \dot{R}}{\partial \sigma_7} = \frac{G}{r^2} \cos \sigma_1$$

$$\frac{\partial \dot{R}}{\partial \sigma_8} = - \frac{2\dot{R}}{r} \frac{\partial r}{\partial \sigma_8}$$

$$f = \frac{r\dot{R} + \dot{R}r}{2G}$$

$$\underline{k = 1, 2, \dots, 8:}$$

$$\frac{\partial f}{\partial \sigma_k} = \frac{1}{2G} \left( \dot{r} \frac{\partial R}{\partial \sigma_k} + R \frac{\partial \dot{r}}{\partial \sigma_k} + \dot{R} \frac{\partial r}{\partial \sigma_k} + r \frac{\partial \dot{R}}{\partial \sigma_k} - 2f \frac{\partial G}{\partial \sigma_k} \right)$$